Goldstein
9.21 (a)

$$H = \frac{p^2}{2} - \frac{1}{2q^2} \quad , \qquad D = \frac{pq}{2} - Ht$$

$$[H, D] = \frac{\partial H}{\partial q}\frac{\partial D}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial D}{\partial q}$$

$$= \left(\frac{1}{q^3}\right)\left(\frac{q}{2}\right) - (p)\left(\frac{p}{2}\right)$$

$$= \frac{1}{2q^2} - \frac{p^2}{2} = \boxed{-H}$$

evidently, $\frac{\partial D}{\partial t} = -H$ as well, thus D satisfies

$[H, D] = \frac{\partial D}{\partial t}$ and is thus a constant of motion.

Goldstein
9. 21 (b)

$$H = [p_i p_i]^{n/2} - a[r_i r_i]^{-n/2}$$

$$D = \frac{p_i r_i}{n} - Ht$$

$$[H, D] = \frac{\partial H}{\partial r_i}\frac{\partial D}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial D}{\partial r_i}$$

$$\frac{\partial H}{\partial r_i} = (-a)(-n/2)[r_j r_j]^{-n/2-1}\, 2r_i$$

$$= an[r_j r_j]^{-n/2-1}\, r_i$$

$$\frac{\partial H}{\partial p_i} = \frac{n}{2}[p_j p_j]^{n/2-1}\, 2p_i = n[p_j p_j]^{n/2-1}\, p_i$$

$$\frac{\partial D}{\partial p_i} = \frac{r_i}{n}, \quad \frac{\partial D}{\partial r_i} = \frac{p_i}{n}$$

$$\Rightarrow [H, D] = \left[a\cancel{n}[r_j r_j]^{-n/2-1}\, r_i\right]\frac{(r_i)}{\cancel{n}} - \cancel{n}[p_j p_j]^{n/2-1}\, p_i \frac{p_i}{\cancel{n}}$$

$$= \boxed{a[r_j r_j]^{-n/2} - [p_j p_j]^{n/2} = -H = \frac{\partial D}{\partial t}}$$

Goldstein 9.21 (c)

The transformation is defined by

$$q \to Q = \lambda q, \quad p \to P = \lambda^{-1} p$$

The Hamiltonian is

$$K(P, Q) = \frac{\lambda^2 P^2}{2} - \frac{\lambda^2}{2Q^2}$$

It gives eqm

$$\begin{cases} \dot{Q} = \dfrac{\partial K}{\partial P} = \lambda^2 P, \\ \dot{P} = -\dfrac{\partial K}{\partial Q} = -\lambda^2 Q^{-3}, \end{cases}$$

In terms of p, q, they are

$$\begin{cases} \lambda \dot{q} = \lambda^2 \lambda^{-1} p \\ \lambda^{-1} \dot{p} = -\lambda^2 \lambda^{-3} q^{-3} \end{cases} \Leftrightarrow \begin{cases} \dot{q} = p \\ \dot{p} = -q^{-3} \end{cases}$$

which are just eqm in the old Hamiltonian.

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